A note on powers of a group

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Let G be a group and $n_1, n_2, \ldots, n_k$ integers with the greatest common divisor d. The following theorem was proved in [1] and re-proved in [3]:

The power $\{G^d\}$[1]) is cyclic if and only if $\{G^{n_1}\}, \{G^{n_2}\}, \ldots, \{G^{n_k}\}$ are cyclic.

The proof of sufficiency is essentially based on the fact that $\{G^d\}$ should be abelian. It is the purpose of this note to show that this conclusion follows from the mere assumption of commutativity of $\{G^{n_1}\}, \{G^{n_2}\}, \ldots, \{G^{n_k}\}$, thus answering affirmatively the second and, hence, also the first problem stated at the end of [3]. The result can be formulated as follows:

Theorem. *The power $\{G^d\}$ is abelian, or a direct product of cyclic groups, if and only if the powers $\{G^{n_1}\}, \{G^{n_2}\}, \ldots, \{G^{n_k}\}$ are abelian, or direct products of cyclic groups, respectively. Moreover, if — in the latter case — $\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k$ and $\mathfrak{d}$ denote consecutively the minimal cardinalities of the sets of cyclic factors of $\{G^{n_1}\}, \{G^{n_2}\}, \ldots$ $\ldots, \{G^{n_k}\}$ and $\{G^d\}$ in the different decompositions into a direct product of cyclic groups, then $\mathfrak{d} = \max(\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k)$.*

Evidently, the particular case of $\mathfrak{n}_1 = \mathfrak{n}_2 = \ldots = \mathfrak{n}_k = 1$ yields the above mentioned result.

The theorem is an immediate consequence of the following two propositions. In the formulation and the proof of the first one, d stands always for the greatest common divisor of integers n_1, n_2: $n_1 = dm_1, n_2 = dm_2$; hence, integers t_1, t_2 exist such that $d = t_1 n_1 + t_2 n_2$.

Proposition 1. *Let S be a complex in a group G, $N(S)$ its normalizator (in G). Let*

$$S^{n_1} \cup S^{n_2} \subseteqq N(S). \tag{*}$$

If $\{S^{n_1}\}$ and $\{S^{n_2}\}$ are abelian, then also $\{S^d\}$ is abelian.[2])

Proof. Always $\{S^d\} = \{S^{n_1} \cup S^{n_2}\} = \{\{S^{n_1}\}, \{S^{n_2}\}\}$. Since both $\{S^{n_1}\}$ and $\{S^{n_2}\}$ are abelian, the intersection $\{S^{n_1}\} \cap \{S^{n_2}\}$ is contained in the center Z of $\{S^d\}$. Moreover, the condition (*) implies that both $\{S^{n_1}\}$ and $\{S^{n_2}\}$ are normal in $\{S^d\}$.

[1]) I. e. the subgroup generated by G^d — the set of all the elements g^d with $g \in G$.

[2]) The example of the complex of the symmetric group S_3 consisting of the permutations (1, 2) and (1, 2, 3) shows that the condition (*) is essential (taking $n_1 = 2$, $n_2 = 3$).

Now, let a, b be two arbitrary elements of S. The commutator of a^d and b^d belongs to Z: for,

$$a^{-d}b^{-d}a^d b^d = a^{-t_1 n_1}\big(b^{-t_2 n_2}(a^{-t_2 n_2}(a^{t_1 n_1} b^{-t_1 n_1})a^{t_2 n_2})b^{t_2 n_2}\big)b^{t_1 n_1} =$$
$$= \big(a^{-t_1 n_1}(a^{-t_2 n_2} b^{-t_2 n_2})a^{t_1 n_1}\big)\big(b^{-t_1 n_1}(a^{t_2 n_2} b^{t_2 n_2})b^{t_1 n_1}\big)$$

belongs to both $\{S^{n_1}\}$ and $\{S^{n_2}\}$. Thus,

$$(**) \qquad a^d b^d = b^d a^d z$$

for a certain $z \in Z$. But, using (**),

$$a^{n_i} b^{n_i} = b^{n_i} a^{n_i} z^{m_i^2} \quad \text{for} \quad i = 1, 2,$$

and we get

$$z^{m_1^2} = z^{m_2^2} = 1.$$

Hence $z = 1$, i. e. any two elements of S^d commute and $\{S^d\}$ is abelian.

Proposition 2. *Let H be a direct product of cyclic groups and $\mathfrak{n}$ the least cardinality of factors in a decomposition of H into a direct product of cyclic groups. Denote by $\mathfrak{r}_p$ the cardinality of the set of p-primary cyclic factors and by $\mathfrak{r}_0$ the cardinality of the set of all infinite cyclic factors in a decomposition of H into a direct product of indecomposable factors; further, denote by $\mathfrak{r}^*$ the least cardinal number greater or equal than $\mathfrak{r}_p$ for every p. Then*

$$\mathfrak{n} = \mathfrak{r}^* + \mathfrak{r}_0 \ \textit{ if } \ \mathfrak{r}^* + \mathfrak{r}_0 \geqq \aleph_0 \ \textit{ or if } \ \mathfrak{r}_p = 0 \ \textit{ for almost all } p, \text{[3]}$$

and $\mathfrak{n} = \aleph_0$ otherwise.

Proof. The statement follows at once from the fact that a direct product of two cyclic groups is cyclic if and only if their orders are (finite and) relatively prime.

Proof of the Theorem. Applying Proposition 1, the first part of the theorem can easily be proved by induction, since the condition (*) is satisfied in a trivial way. Hence,

$$(***) \qquad \{G^d\}^{n_i/d} = \{G^{n_i}\} \quad \text{for} \quad i = 1, 2, \ldots, k.$$

The assertion on direct products of cyclic groups is then a consequence of Theorem 12.4 of [2]. Finally, making use of the relations (***) and Proposition 2 we deduce the second part of the theorem, thus completing the proof.

Bibliography

[1] V. Dlab, On cyclic groups, *Czech. Math. J.*, **10/85** (1960), 244—254.
[2] L. Fuchs, *Abelian groups* (Budapest, 1958).
[3] F. Szász, Bemerkung zu meiner Arbeit „Über Gruppen, deren sämtliche nichttriviale Potenzen zyklische Untergruppen der Gruppe sind", *Acta Sci. Math.*, **23** (1962), 64—66.

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[3]) I. e. $\mathfrak{n}$ is equal to the reduced rank of H in this case (cf. [2]).